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Simultaneous (C, A, B)-pairs for Infinite-Dimensional Systems*

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In this paper, the infinite-dimensional version of the simultaneous (C, A, B)-pair is introduced and its properties are investigated. And robust disturbance-rejection problem with dynamic compensator is formulated and then its solvability conditions are presented. © 1999 Academic Press

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1. INTRODUCTION

In the framework of the so-called geometric approach, many control problems with state feedback or incomplete-state feedback (e.g., decoupling problems, disturbance-rejection problems, etc.) have been studied for finite-dimensional systems (see e.g., [14]). Further, the notion of (C, A, B)-pair was first introduced by Schumacher [13] and this concept has been used successfully to design dynamic compensators. After that,

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Curtain extended the geometric concepts to infinite-dimensional systems and various control problems were studied (see e.g., [1–3], [6–9]). On the other hand, from the practical viewpoint Ghosh studied the concepts of simultaneous invariant subspaces and simultaneous (C, A, B)-pair, and the robust disturbance-rejection problems for uncertain linear systems in the sense that systems matrices are represented as convex combinations were studied [4, 10]. Further, the present authors extended the concept of simultaneous invariance to infinite-dimensional systems, and the robust disturbance-rejection problems with state feedback and with incomplete-state feedback were studied [11, 12].

The objective of this paper is to investigate the notion of simultaneous (C, A, B)-pair for infinite-dimensional systems and to study the robust disturbance-rejection problem with dynamic compensator.

This paper is organized as follows. Section 2 will give the concept of simultaneous (C, A, B)-pair and its properties. In Sect. 3, the robust disturbance-rejection problem with dynamic compensator is formulated and its solvability conditions will be presented. Finally, Sect. 4 will give some concluding remarks.

2. SIMULTANEOUS (C, A, B)-PAIRS

First, we give some notations used throughout this investigation. Let $\mathbf{B}(X; Y)$ denote the set of all bounded linear operators from a Hilbert space X into another Hilbert space Y; for notational simplicity, we write $\mathbf{B}(X)$ for $\mathbf{B}(X; X)$. For a linear operator A the domain, the image, the kernel, the resolvent set, the inverse operator and the C_0-semigroup generated by A are denoted by $\mathrm{D}(A)$, $\mathrm{Im}\, A$, $\mathrm{Ker}\, A$, $\rho(A)$, A^{-1} and $\{S_A(t);\ t \geq 0\}$, respectively. Further, the dimension and the orthogonal complement of a closed subspace V are denoted by $\dim(V)$ and $(V)^{\perp}$, respectively, and an identity operator by I. For a positive integer r we use the notation $\mathbf{r} := \{1, \dots, r\}$.

Next, consider the family $\{\Sigma_{ijk};\ i \in \mathbf{r}_1,\ j \in \mathbf{r}_2,\ k \in \mathbf{r}_3\}$ of $r_1 \times r_2 \times r_3$ systems defined in a Hilbert space X:

$$\Sigma_{ijk}: \begin{cases} \dfrac{d}{dt} x(t) = A_i x(t) + B_j u(t), \\ y(t) = C_k x(t), (i \in \mathbf{r}_1, j \in \mathbf{r}_2, k \in \mathbf{r}_3) \end{cases} \tag{1}$$

where A_i is the infinitesimal generator of a C_0-semigroup $\{S_{A_i}, (t);\ t \geq 0\}$ on X, while B_j is a bounded linear operator from Euclidean space $\mathbf{R}^m$ into X (i.e., $B_j \in \mathbf{B}(\mathbf{R}^m; X)$, $j \in \mathbf{r}_2$), C_k is a bounded linear operator from X

into $\mathbf{R}^p$ (i.e., $C_k \in \mathbf{B}(X; \mathbf{R}^p)$, $k \in \mathbf{r}_3$) and $x(t) \in X$, $u(t) \in \mathbf{R}^m$, $y(t) \in \mathbf{R}^p$ are the state, the input and the measurement output, respectively.

Now, introduce a compensator (K, L, M, N) defined on a Hilbert space W of the following form:

$$\begin{aligned} \frac{d}{dt} w(t) &= Nw(t) + My(t), \\ u(t) &= Lw(t) + Ky(t), \end{aligned} \tag{2}$$

where N is the infinitesimal generator of a C_0-semigroup $\{S_N(t);\ t \geq 0\}$ on a Hilbert space W, $M \in \mathbf{B}(\mathbf{R}^p; W)$, $L \in \mathbf{B}(W; \mathbf{R}^m)$ and $K \in \mathbf{B}(\mathbf{R}^p; \mathbf{R}^m)$.

If a compensator of the form (2) is applied to system (1), the resulting closed-loop system with the extended state space $X^e := X \oplus W$ is easily seen to be

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ w(t) \end{bmatrix} = \begin{bmatrix} A_i + B_j K C_k & B_j L \\ M C_k & N \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \end{bmatrix}, \tag{3}$$

where $X \oplus W$ means the direct sum of X and W. For the combined system (3), define

$$x^e(t) := \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} \quad \text{and} \quad A^e_{ijk} := \begin{bmatrix} A_i + B_j K C_k & B_j L \\ M C_k & N \end{bmatrix} \tag{4}$$

with domain $\mathrm{D}(A^e_{ijk})$ $(= \mathrm{D}(A_i) \oplus W)$.

For a family $\{\Sigma_{ijk};\ i \in \mathbf{r}_1,\ j \in \mathbf{r}_2,\ k \in \mathbf{r}_3\}$ of systems, we give the following invariant subspaces.

DEFINITION 2.1. Let V be a closed subspace of X.

(i) V is said to be $\{(A_i, B_j);\ i \in \mathbf{r}_1,\ j \in \mathbf{r}_2\}$-invariant if there exists an $F \in \mathbf{B}(X; \mathbf{R}^m)$ such that

$$(A_i + B_j F)(V \cap \mathrm{D}(A_i)) \subset V \text{ for all } i \in \mathbf{r}_1, j \in \mathbf{r}_2.$$

$\mathbf{F}(V) := \{F \in \mathbf{B}(X; \mathbf{R}^m) | (A_i + B_j F)(V \cap \mathrm{D}(A_i)) \subset V$ for all $i \in \mathbf{r}_1,\ j \in \mathbf{r}_2\}$.

(ii) V is said to be $\{S(A_i, B_j);\ i \in \mathbf{r}_1,\ j \in \mathbf{r}_2\}$-invariant if there exists an $F \in \mathbf{B}(X; \mathbf{R}^m)$ such that

$$S_{A_i + B_j F}(t) V \subset V \quad \text{for all } t \geq 0 \text{ and all } i \in \mathbf{r}_1, j \in \mathbf{r}_2.$$

$\mathbf{F}_s(V) := \{F \in \mathbf{B}(X; \mathbf{R}^m) | S_{A_i + B_j F}(t) V \subset V$ for all $t \geq 0$ and all $i \in \mathbf{r}_1,\ j \in \mathbf{r}_2\}$.

(iii) V is said to be $\{(C_k, A_i); i \in \mathbf{r}_1, k \in \mathbf{r}_3\}$-invariant if there exists a $G \in \mathbf{B}(\mathbf{R}^p; X)$ such that

$$(A_i + GC_k)(V \cap \mathrm{D}(A_i)) \subset V \quad \text{for all } i \in \mathbf{r}_1, k \in \mathbf{r}_3.$$

$\mathbf{G}(V) := \{G \in \mathbf{B}(\mathbf{R}^p; X) | (A_i + GC_k)(V \cap \mathrm{D}(A_i)) \subset V$ for all $i \in \mathbf{r}_1$, $k \in \mathbf{r}_3\}$.

(iv) V is said to be $\{S(C_k, A_i); i \in \mathbf{r}_1, k \in \mathbf{r}_3\}$-invariant if there exists a $G \in \mathbf{B}(\mathbf{R}^p; X)$ such that

$$S_{A_i + GC_k}(t)V \subset V \quad \text{for all } t \geq 0 \text{ and all } i \in \mathbf{r}_1, k \in \mathbf{r}_3.$$

$\mathbf{G}_s(V) := \{G \in \mathbf{B}(\mathbf{R}^p; X) | S_{A_i + GC_k}(t)V \subset V$ for all $t \geq 0$ and all $i \in \mathbf{r}_1, k \in \mathbf{r}_3\}$.

Remark 2.2. (i) For a family $\{\Sigma_{ijk}; i \in \mathbf{r}_1, j \in \mathbf{r}_2, k \in \mathbf{r}_3\}$ of systems, an $\{S(A_i, B_j); i \in \mathbf{r}_1, j \in \mathbf{r}_2\}$-invariant subspace V has the property that if an arbitrary initial state $x(0) \in V$ then there exists a state feedback $u(t) = Fx(t)$ which is independent of i and j such that state trajector $x(t) \in V$ for all $t \geq 0$.

(ii) If A_i $(i \in \mathbf{r}_1)$ are bounded linear operators on X (i.e., $A_i \in \mathbf{B}(X)$), then the statements (i) and (ii), and (iii) and (iv) in Definition 2.1 are equivalent, respectively

LEMMA 2.3. (i) *If V is $\{S(A_i, B_j); i \in \mathbf{r}_1, j \in \mathbf{r}_2\}$-invariant, then* $\mathbf{F}(V) = \mathbf{F}_s(V)$.

(ii) *If V is $\{S(C_k, A_i); i \in \mathbf{r}_1, k \in \mathbf{r}_3\}$-invariant, then* $\mathbf{G}(V) = \mathbf{G}_s(V)$.

Proof. The proof follows from [2]. ∎

For finite-dimensional systems, Schumacher [13] first introduced the concept of (C, A, B)-pair, and Ghosh [4] and Curtain [1] extended this concept to simultaneous version of finite-dimensional systems and infinite-dimensional version, respectively. The following definition is a simultaneous and infinite-dimensional version of (C, A, B)-pair.

DEFINITION 2.4. Let V_1 and V_2 be closed subspaces of X. A pair (V_1, V_2) of subspaces is said to be $\{S(C_k, A_i, B_j); i \in \mathbf{r}_1, j \in \mathbf{r}_2, k \in \mathbf{r}_3\}$-pair if the following three conditions hold.

(i) V_1 is $\{S(C_k, A_i); i \in \mathbf{r}_1, k \in \mathbf{r}_3\}$-invariant.

(ii) V_2 is $\{S(A_i, B_j); i \in \mathbf{r}_1, j \in \mathbf{r}_2\}$-invariant.

(iii) $V_1 \subset V_2$.

For closed-loop system (3) with (4), we give the following definition.

DEFINITION 2.5. Let V^e be closed subspace of X^e.

(i) V^e is said to be an $\{A^e_{ijk};\ i \in \mathbf{r}_1,\ j \in \mathbf{r}_2,\ k \in \mathbf{r}_3\}$-invariant if $A^e_{ijk}(V^e \cap \mathrm{D}(A^e_{ijk})) \subset V^e$ for all $i \in \mathbf{r}_1$, $j \in \mathbf{r}_2$, $k \in \mathbf{r}_3$.

(ii) V^e is said to be an $\{S_{A^e_{ijk}}(t);\ i \in \mathbf{r}_1,\ j \in \mathbf{r}_2,\ k \in \mathbf{r}_3\}$-invariant if $S_{A^e_{ijk}}(t)V^e \subset V^e$ for all $t \geq 0$ and all $i \in \mathbf{r}_1$, $j \in \mathbf{r}_2$, $k \in \mathbf{r}_3$.

The following lemma was obtained by Zwart.

LEMMA 2.6 [15]. *Let V^e be closed subspace of X^e and the following three subspaces are introduced*:

$$S_{\mathrm{orth}} := \left\{ x \in X \left| \begin{bmatrix} x \\ w \end{bmatrix} \in [V^e]^{\perp} \text{ for some } w \in W \right. \right\} = P_X\left([V^e]^{\perp}\right),$$

$$S_1 := [S_{\mathrm{orth}}]^{\perp},$$

$$S_2 := \left\{ x \in X \left| \begin{bmatrix} x \\ w \end{bmatrix} \in [V^e] \text{ for some } w \in W \right. \right\} = P_X([V^e]),$$

where P_X is the projection operator from X^e onto X along W. Then, the following statements hold.

(i) $S_1 = \{x \in X \,|\, \left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right] \in V^e\}$.

(ii) $S_1 \subset S_2$.

(iii) *If* $\dim(W) < \infty$, *then S_2 is a closed subspace of X and* $\dim(S_2 \cap S_1^{\perp}) < \infty$.

The following lemma can be easily obtained.

LEMMA 2.7. *If $V^e (\subset X^e)$ is an $\{A^e_{ijk};\ i \in \mathbf{r}_1,\ j \in \mathbf{r}_2,\ k \in \mathbf{r}_3\}$-invariant, then*

$$(A_{i_1} - A_{i_2})\left(S_2 \cap \mathrm{D}(A_{i_1}) \cap \mathrm{D}(A_{i_2})\right) \subset S_1 \quad \text{for all } i_1, i_2 \in \mathbf{r}_1.$$

Proof. Suppose that V^e is an $\{A^e_{ijk};\ i \in \mathbf{r}_1,\ j \in \mathbf{r}_2,\ k \in \mathbf{r}_3\}$-invariant and i_1, i_2 are arbitrary elements of $\mathbf{r}_1$. Let x be an arbitrary element of $(S_2 \cap \mathrm{D}(A_{i_1}) \cap \mathrm{D}(A_{i_2}))$. Then, $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in (V^e \cap \mathrm{D}(A^e_{i_1jk}) \cap \mathrm{D}(A^e_{i_2jk}))$ for some $w \in W$. Since V^e is $\{A^e_{ijk};\ i \in \mathbf{r}_1,\ j \in \mathbf{r}_2,\ k \in \mathbf{r}_3\}$-invariant,

$$A^e_{i_1jk}\begin{bmatrix} x \\ w \end{bmatrix} \in V^e \quad \text{and} \quad A^e_{i_2jk}\begin{bmatrix} x \\ w \end{bmatrix} \in V^e.$$

Hence,

$$\left(A^e_{i_1jk} - A^e_{i_2jk}\right)\begin{bmatrix} x \\ w \end{bmatrix} = \left(\begin{bmatrix} A_{i_1} + B_jKC_k & B_jL \\ MC_k & N \end{bmatrix} - \begin{bmatrix} A_{i_2} + B_jKC_k & B_jL \\ MC_k & N \end{bmatrix}\right)\begin{bmatrix} x \\ w \end{bmatrix}$$

$$= \begin{bmatrix} A_{i_1} - A_{i_2} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ w \end{bmatrix}$$

$$= \begin{bmatrix} \left(A_{i_1} - A_{i_2}\right)x \\ 0 \end{bmatrix} \in V^e.$$

It follows from Lemma 2.6(i) that $(A_{i_1} - A_{i_2})x \in S_1$, showing that

$$\left(A_{i_1} - A_{i_2}\right)\left(S_2 \cap \mathrm{D}\left(A_{i_1}\right) \cap \mathrm{D}\left(A_{i_2}\right)\right) \subset S_1 \quad \text{for all } i_1, i_2 \in \mathbf{r}_1.$$

∎

Let ρ^i_∞ denote the largest connected subset of $\rho(A_i)$ that contains an interval of the form $[\tau_i, +\infty]$ for some $\tau_i \in \mathbf{R}$. Since A_i generates a C_0-semigroup, ρ^i_∞ is nonempty [15]. Therefore, for a sufficiently large $s \in \mathbf{R}$, $s \in \bigcap_{i=1}^{r_1} \rho^i_\infty$.

The following lemmas play important role to prove our main results.

LEMMA 2.8. *Let V be a closed subspace of X. Then, the following three statements are equivalent.*

(i) $S_{A_i}(t)V \subset V$ *for all* $t \geq 0$ *and all* $i \in \mathbf{r}_1$.

(ii) $(sI - A_i)^{-1}V \subset V$ *for an* $s \in \bigcap_{i=1}^{r_1} \rho^i_\infty$ *and all* $i \in \mathbf{r}_1$.

(iii) $(sI - A_i)^{-1}V \subset V$ *for all* $s \in \bigcap_{i=1}^{r_1} \rho^i_\infty$ *and all* $i \in \mathbf{r}_1$.

Proof. The proofs follow from [15]. ∎

LEMMA 2.9 [1, 3, 15]. *Suppose that A is the infinitesimal generator of a C_0-semigroup $\{S_A(t);\, t \geq 0\}$ on X, V is a closed subspace of X and $Q_1 \in \mathbf{B}(X)$. Then, the following statements hold.*

(i) *If* $S_A(t)V \subset V$ *for all* $t \geq 0$, *then* $A(V \cap \mathrm{D}(A)) \subset V$.

(ii) *If* $V \subset \mathrm{D}(A)$ *and* $AV \subset V$, *then* $S_A(t)V \subset V$ *for all* $t \geq 0$.

(iii) *If* $S_{A+Q_1}(t)V \subset V$ *for all* $t \geq 0$, *then* $\overline{V \cap \mathrm{D}(A)} = V$.

(iv) *If there exists a* $Q_2 \in \mathbf{B}(X)$ *such that* $S_{A+Q_2}(t)V \subset V$ *for all* $t \geq 0$ *and* $(Q_1 - Q_2)(V \cap \mathrm{D}(A)) \subset V$, *then* $S_{A+Q_1}(t)V \subset V$ *for all* $t \geq 0$.

(v) *If there exists a* $Q_2 \in \mathbf{B}(X)$ *such that* $S_{A+Q_2}(t)V \subset V$ *for all* $t \geq 0$ *and* $(Q_1 - Q_2)(V \cap \mathrm{D}(A)) = \{\mathbf{0}\}$, *then* $S_{A+Q_1}(t)x = S_{A+Q_2}(t)x$ *for all* $t \geq 0$ *and all* $x \in V$.

LEMMA 2.10 [9]. *Let* U_1 *and* U_2 *be real Hilbert spaces, and* $F_i \in \mathbf{B}(X; U_i)$ $(i = 1, 2)$ *be given. If* $\operatorname{Im} F_2$ *is closed in* U_2, *then the following statements are equivalent.*

(i) $\operatorname{Ker} F_1 \supset \operatorname{Ker} F_2$.

(ii) *There exists a* $K \in \mathbf{B}(U_2; U_1)$ *such that* $F_1 = KF_2$.

In the following discussion, it is assumed that $\mathbf{r}_1 = \mathbf{r}_2 = \mathbf{r}_3 = \mathbf{2}$ for system (1).

The following lemma is extensions of the results of Curtain [1] to simultaneous version and of the results of Ghosh [4] to infinite-dimensional systems.

LEMMA 2.11. *Suppose that system* (1) *satisfies* $\mathrm{D}(A_1) \subset \mathrm{D}(A_2)$. *If a pair* (V_1, V_2) *of subspaces of* X *is an* $\{S(C_k, A_i, B_j);\ i, j, k \in \mathbf{2}\}$*-pair, then there exist* $G \in \mathbf{G}_s(V_1)$, $G_j \in \mathbf{B}(\mathbf{R}^p; X)$, $F_k \in \mathbf{F}_s(V_2)$, $F_0 \in \mathbf{B}(X; \mathbf{R}^m)$ *and* $K \in \mathbf{B}(\mathbf{R}^p; \mathbf{R}^m)$ *such that*

$$\begin{aligned} G &= B_j K + G_j, \quad \operatorname{Im} G_j \subset V_2 \\ F_k &= KC_k + F_0, \quad \operatorname{Ker} F_0 \supset V_1 \quad \textit{for all } j, k \in \mathbf{2}. \end{aligned} \tag{5}$$

Proof. Suppose that a pair (V_1, V_2) is an $\{S(C_k, A_i, B_j);\ i, j, k \in \mathbf{2}\}$-pair. Then, since V_1 is $\{S(C_k, A_i);\ i, k \in \mathbf{2}\}$-invariant, by Lemma 2.3(ii), choose a $G_0 \in \mathbf{G}(V_1) = \mathbf{G}_s(V_1)$. Let $x \in V_1$ be an arbitrary element. Then, by Lemma 2.9(iii), there exists a sequence $x_n \in (V_1 \cap \mathrm{D}(A_1))$ such that $\lim_{n\to\infty} x_n = x$. Moreover, since V_2 is $\{S(A_i, B_j);\ i, j \in \mathbf{2}\}$-invariant, we can easily obtain

$$\begin{bmatrix} A_1 \\ A_2 \\ A_1 \\ A_2 \end{bmatrix} \left(V_2 \in \mathrm{D}(A_1)\right) \subset \begin{bmatrix} V_2 \\ V_2 \\ V_2 \\ V_2 \end{bmatrix} + \operatorname{Im} \begin{bmatrix} B_1 \\ B_1 \\ B_2 \\ B_2 \end{bmatrix}. \tag{6}$$

Then, we have

$$\begin{aligned}\begin{bmatrix} G_0C_k \\ G_0C_k \\ G_0C_k \\ G_0C_k \end{bmatrix} x_n &= \begin{bmatrix} (A_1 + G_0C_k)x_n \\ (A_2 + G_0C_k)x_n \\ (A_1 + G_0C_k)x_n \\ (A_2 + G_0C_k)x_n \end{bmatrix} - \begin{bmatrix} A_1x_n \\ A_2x_n \\ A_1x_n \\ A_2x_n \end{bmatrix} \\ &\in \begin{bmatrix} V_1 \\ V_1 \\ V_1 \\ V_1 \end{bmatrix} + \begin{bmatrix} A_1 \\ A_2 \\ A_1 \\ A_2 \end{bmatrix} \big(V_1 \cap \mathrm{D}(A_1)\big) \\ &\subset \begin{bmatrix} V_2 \\ V_2 \\ V_2 \\ V_2 \end{bmatrix} + \operatorname{Im} \begin{bmatrix} B_1 \\ B_1 \\ B_2 \\ B_2 \end{bmatrix} \quad \big(\text{by } V_1 \subset V_2 \text{ and } (6)\big).\end{aligned}$$

Since $\dim\left\{\operatorname{Im}\begin{bmatrix} B_1 \\ B_1 \\ B_2 \\ B_2 \end{bmatrix}\right\} < \infty$, it remarks that $\begin{bmatrix} V_2 \\ V_2 \\ V_2 \\ V_2 \end{bmatrix} + \operatorname{Im}\begin{bmatrix} B_1 \\ B_1 \\ B_2 \\ B_2 \end{bmatrix}$ is a closed subspace. By the boundedness of $\begin{bmatrix} G_0C_k \\ G_0C_k \\ G_0C_k \\ G_0C_k \end{bmatrix}$ and the closedness of $\begin{bmatrix} V_2 \\ V_2 \\ V_2 \\ V_2 \end{bmatrix} + \operatorname{Im}\begin{bmatrix} B_1 \\ B_1 \\ B_2 \\ B_2 \end{bmatrix}$, we obtain

$$\begin{bmatrix} G_0C_k \\ G_0C_k \\ G_0C_k \\ G_0C_k \end{bmatrix} x = \lim_{n\to\infty} \begin{bmatrix} G_0C_k \\ G_0C_k \\ G_0C_k \\ G_0C_k \end{bmatrix} x_n \in \begin{bmatrix} V_2 \\ V_2 \\ V_2 \\ V_2 \end{bmatrix} + \operatorname{Im} \begin{bmatrix} B_1 \\ B_1 \\ B_2 \\ B_2 \end{bmatrix},$$

showing that

$$\begin{bmatrix} G_0C_k \\ G_0C_k \\ G_0C_k \\ G_0C_k \end{bmatrix} V_1 \subset \begin{bmatrix} V_2 \\ V_2 \\ V_2 \\ V_2 \end{bmatrix} + \operatorname{Im} \begin{bmatrix} B_1 \\ B_1 \\ B_2 \\ B_2 \end{bmatrix} \quad (k \in \mathbf{2}). \tag{7}$$

Let x be an element of $\mathbf{R}^p = (C_1V_1 + C_2V_1) \oplus (C_1V_1 + C_2V_1)^\perp$. Then, x can be decomposed by $x = y + z$, where $y \in (C_1V_1 + C_2V_1)$, $z \in (C_1V_1 + C_2V_1)^\perp$. Define $G \in \mathbf{B}(\mathbf{R}^p; X)$ by $Gx := G_0 y$. Then, since $GC_k - G_0C_k)V_1 = \{\mathbf{0}\}$, it follows from Lemma 2.9(v) that $G \in \mathbf{G}_s(V_1)$.

Now, let η be an arbitrary element of $\operatorname{Im}\begin{bmatrix} G \\ G \\ G \\ G \end{bmatrix}$. Then, there exists $\xi \in \mathbf{R}^p$ such that

$$\eta = \begin{bmatrix} G \\ G \\ G \\ G \end{bmatrix} \xi \quad \text{and} \quad \xi = \xi_1 + \xi_2,$$

where $\xi_1 \in (C_1 V_1 + C_2 V_1)$, $\xi_2 \in (C_1 V_1 + C_2 V_1)^{\perp}$ .Then,

$$\eta = \begin{bmatrix} G \\ G \\ G \\ G \end{bmatrix} \xi = \begin{bmatrix} G_0 \xi_1 \\ G_0 \xi_1 \\ G_0 \xi_1 \\ G_0 \xi_1 \end{bmatrix}. \tag{8}$$

Moreover, ξ_1 is represented by $\xi_1 = C_1 x_1 + C_2 x_2$ for some $x_1, x_2 \in V_1$. Hence, by (7) and (8) we obtain

$$\eta = \begin{bmatrix} G_0 \xi_1 \\ G_0 \xi_1 \\ G_0 \xi_1 \\ G_0 \xi_1 \end{bmatrix} = \begin{bmatrix} G_0 C_1 x_1 \\ G_0 C_1 x_1 \\ G_0 C_1 x_1 \\ G_0 C_1 x_1 \end{bmatrix} + \begin{bmatrix} G_0 C_2 x_2 \\ G_0 C_2 x_2 \\ G_0 C_2 x_2 \\ G_0 C_2 x_2 \end{bmatrix} \in \begin{bmatrix} V_2 \\ V_2 \\ V_2 \\ V_2 \end{bmatrix} + \operatorname{Im}\begin{bmatrix} B_1 \\ B_1 \\ B_2 \\ B_2 \end{bmatrix}$$

which imply

$$\operatorname{Im}\begin{bmatrix} G \\ G \\ G \\ G \end{bmatrix} \subset \begin{bmatrix} V_2 \\ V_2 \\ V_2 \\ V_2 \end{bmatrix} + \operatorname{Im}\begin{bmatrix} B_1 \\ B_1 \\ B_2 \\ B_2 \end{bmatrix}. \tag{9}$$

Next, let $\{y_1, \ldots, y_p\}$ be a basis of $\mathbf{R}^p$. Then, by (9) there exist $\{x_1^i, x_2^i, x_3^i, x_4^i\} \in V_2$ and $u^i \in \mathbf{R}^m$ such that

$$\begin{bmatrix} G \\ G \\ G \\ G \end{bmatrix} y_i = \begin{bmatrix} x_1^i \\ x_2^i \\ x_3^i \\ x_4^i \end{bmatrix} + \begin{bmatrix} B_1 \\ B_1 \\ B_2 \\ B_2 \end{bmatrix} u^i. \tag{10}$$

Let us define K: $\mathbf{R}^p \to \mathbf{R}^m$, G_j: $\mathbf{R}^p \to X$ $(j = 1, 2)$ by

$$Ky_i := u_i \qquad (i = 1, \dots, p),$$

$$G_1 y_i := x_1^i \in V_2 \qquad (i = 1, \dots, p), \tag{11}$$

$$G_2 y_i := x_3^i \in V_2 \qquad (i = 1, \dots, p.) \tag{12}$$

Then, by (10) we obtain

$$\begin{bmatrix} G \\ G \\ G \\ G \end{bmatrix} y_i = \begin{bmatrix} G_1 y_i \\ x_2^i \\ G_2 y_i \\ x_4^i \end{bmatrix} + \begin{bmatrix} B_1 \\ B_1 \\ B_2 \\ B_2 \end{bmatrix} Ky_i \qquad (i = 1, \dots, p), \tag{13}$$

showing that

$$G = G_1 + B_1 K, \quad G = G_2 + B_2 K. \tag{14}$$

Moreover, (11) and (12) imply

$$\operatorname{Im} G_j \subset V_2 \qquad (j = 1, 2). \tag{15}$$

Now, by (9) there exists $F^\dagger \in \mathbf{B}(X; \mathbf{R}^m)$ such that

$$\operatorname{Im} \begin{bmatrix} GC_k + B_1 F^\dagger \\ GC_k + B_1 F^\dagger \\ GC_k + B_2 F^\dagger \\ GC_k + B_2 F^\dagger \end{bmatrix} \subset \begin{bmatrix} V_2 \\ V_2 \\ V_2 \\ V_2 \end{bmatrix}. \tag{16}$$

Choose an $F^* \in \mathbf{F}_s(V_2)$ $(= \mathbf{F}(V_2))$ and define $F_0 \in \mathbf{B}(X; \mathbf{R}^m)$ by

$$F_0 := \begin{cases} F^* + F^\dagger & \text{on } V_1^\perp, \\ 0 & \text{on } V_1. \end{cases}$$

Then, clearly $V_1 \subset \operatorname{Ker} F_0$. Moreover, we obtain

$$\big(A_i + B_j F_0 + GC_k\big)\big(V_2 \cap \mathrm{D}(A_1)\big) \subset V_2 \quad \text{for all } i, j, k \in \mathbf{2}. \tag{17}$$

In fact, if we show that

$$S_{A_i + B_j F_0 + GC_k}(t) V_2 \subset V_2 \quad \text{for all } t \geq 0 \text{ and all } i, j, k \in \mathbf{2}, \tag{18}$$

(17) follows from Lemma 2.9(i). So, we show (18).

Since $S_{A_i+B_jF^*}(t)V_2 \subset V_2$ for all $t \geq 0$ and all $i, j \in \mathbf{2}$, from Lemma 2.9(iv), it suffices to show that

$$\left(B_jF_0 + GC_k - B_jF^*\right)V_2 \subset V_2. \tag{19}$$

In order to prove (19), let $x \in V_2$ be an arbitrary element. Then, x is decomposed as $x = y + z$, $y \in V_1$, $z \in V_1^{\perp}$. Hence,

$$\begin{aligned}\left(B_jF_0 + GC_k - B_jF^*\right)x &= \left(B_jF_0 + GC_k - B_jF^*\right)y \\ &\quad + \left(B_jF_0 + GC_k - B_jF^*\right)z \\ &= \left(GC_k - B_jF^*\right)y + \left(B_jF_0 + GC_k - B_jF^*\right)z.\end{aligned} \tag{20}$$

Now, by Lemma 2.9(iii) there exists a sequence $\{y_n\} \subset (V_1 \cap \mathrm{D}(A_1))$ such that $\lim_{n\to\infty} y_n = y$. Then, since $(GC_k - B_jF^*)$ is a bounded linear operator, $V_1 \subset V_2$, $G \in \mathbf{G}_s(V_1)$ and V_2 is a closed subspace, the first term of (20) satisfies

$$\begin{aligned}\left(GC_k - B_jF^*\right)y &= \lim_{n\to\infty}\left(GC_k - B_jF^*\right)y_n \\ &= \lim_{n\to\infty}\left\{(A_i + GC_k)y_n - \left(A_i + B_jF^*\right)y_n\right\} \\ &\in V_2.\end{aligned} \tag{21}$$

On the other hand, the second term of (20) satisfies

$$\begin{aligned}\left(B_jF_0 + GC_k - B_jF^*\right)z &= \left(B_jF^* + B_jF^{\dagger} + GC_k - B_jF^*\right)z \\ &= \left(GC_k + B_jF^{\dagger}\right)z \\ &\in V_2 \quad (\text{by } (16)).\end{aligned} \tag{22}$$

Hence, it follows from (20)–(22) that

$$\left(B_jF_0 + GC_k - B_jF^*\right)x \in V_2,$$

showing (19). Thus, (17) was proved. Finally, define $F_k \in \mathbf{B}(X; \mathbf{R}^m)$ $(k = 1, 2)$ by $F_k := KC_k + F_0$. Then, $F_k \in \mathbf{F}_s(V_2)$. In fact, let $x \in (V_2 \cap \mathrm{D}(A_1))$ be an arbitrary element. Then, we obtain

$$\begin{aligned}\left(A_i + B_jF_k\right)x &= \left(A_i + B_jKC_k + B_jF_0\right)x \\ &= \left(A_i + B_jF_0 + \left(B_jK + G_j\right)C_k - G_jC_k\right)x \\ &= \left(A_i + B_jF_0 + GC_k\right)x - G_jC_kx \\ &= V_2 + \operatorname{Im} G_j = V_2 \\ &\text{for all } i, j, k \in \mathbf{2} \quad (\text{by } (14), (15), \text{and } (17)).\end{aligned} \tag{23}$$

This completes the proof of Lemma 2.11. ∎

The following two propositions are main results in this section and are used in the next section.

PROPOSITION 2.12. *Suppose that system* (1) *satisfies* $\mathrm{D}(A_1) \subset \mathrm{D}(A_2)$. *If a pair* (V_1, V_2) *of subspaces of* X *is an* $\{S(C_k, A_i, B_j);\ i, j, k \in \mathbf{2}\}$*-pair satisfying* $\mathrm{Im}(B_1 - B_2) \subset V_1, V_2 \subset \{\mathrm{D}(A_1) \cap \mathrm{Ker}(C_1 - C_2)\}$ *and* $(A_1 - A_2)V_2 \subset V_1$, *then there exists a compensator* (K, L, M, N) *on* $\tilde{W} := (V_2 \cap V_1^{\perp})$ *and an* $\{S_{A^e_{ijk}}(t);\ i, j, k \in \mathbf{2}\}$*-invariant subspace* $V^e (\subset X^e)$ *satisfying* $V_1 = S_1$ *and* $V_2 = S_2$, *where* $X^e := X \oplus \tilde{W}$, *and* S_1 *and* S_2 *are defined in Lemma* 2.6.

Proof. Suppose that a pair (V_1, V_2) of subspaces is an $\{S(C_k, A_i, B_j);\ i, j, k \in \mathbf{2}\}$-pair satisfying $\mathrm{Im}(B_1 - B_1) \subset V_1, V_2 \subset \{\mathrm{D}(A_1) \cap \mathrm{Ker}(C_1 - C_2)\}$ and $(A_1 - A_2)V_2 \subset V_1$. Since V_1 and V_2 are closed subspaces and $V_1 \subset V_2$, we have $V_2 = V_1 \oplus (V_2 \cap V_1^{\perp})$. Let $\tilde{W} := (V_2 \cap V_1^{\perp})$ and define $X^e := X \oplus \tilde{W}$. Let R: $V_2 \to \tilde{W}$ be a bounded linear operator such that $\mathrm{Ker} R = V_1$ and $\mathrm{Im}\, R = \tilde{W}$. Then, there exists an operator $R^{\dagger} \in \mathbf{B}(\tilde{W}; V_2)$ such that $RR^{\dagger} = I$ and $R^{\dagger}$ satisfies $R^{\dagger}Rx = 0$ if and only if $x \in V_1$ (see [15, p. 95]). Define the subspace V^e of X^e by

$$V^e := \left\{ \begin{bmatrix} x \\ Rx \end{bmatrix} \middle| x \in V_2 \right\}.$$

Then, $V_1 = S_1$ and $V_2 = S_2$.

Now, it follows from Lemma 2.11 that there exists $G \in \mathbf{G}_s(V_1)$, $G_j \in \mathbf{B}(\mathbf{R}^p; X)$, $F_k \in \mathbf{F}_s(V_2)$, $F_0 \in \mathbf{B}(X; \mathbf{R}^m)$ and $K \in \mathbf{B}(\mathbf{R}^p; \mathbf{R}^m)$ such that

$$G = B_j K + G_j, \quad \mathrm{Im}\, G_j \subset V_2$$

$$F_k = KC_k + F_0, \quad \mathrm{Ker}\, F_0 \supset V_1 \quad \text{for all } j, k \in \mathbf{2}.$$

Define $L \in \mathbf{B}(\tilde{W}; \mathbf{R}^m)$ and $M \in \mathbf{B}(\mathbf{R}^p; \tilde{W})$ by

$$L := F_0 R^{\dagger}, \quad M := -RG_1.$$

Now, we obtain

$$\begin{aligned}(A_i + B_j F_0 + GC_k)V_1 &= (A_i + GC_k)V_1 \\ &\subset V_1 \quad \text{for all } i, j, k \in \mathbf{2}. \end{aligned} \tag{24}$$

Moreover, from (17) in Lemma 2.11, we obtain

$$(A_i + B_j F_0 + GC_k)V_2 \subset V_2 \quad \text{for all } i, j, k \in \mathbf{2}. \tag{25}$$

Then, for an element $x \in V_2$ we have

$$\begin{aligned}(A_1 + B_1 F_1 + G_1 C_1)x &= (A_1 + B_1 K C_1 + B_1 F_0 + G_1 C_1)x \\ &= (A_1 + B_1 F_0 + GC_1)x \\ &\in V_2 \quad (\text{by } (25)). \end{aligned} \tag{26}$$

By (26), we can define $R(A_1 + B_1F_1 + G_1C_1)|_{V_2}: V_2 \to \tilde{W}$. Then,

$$\operatorname{Ker} R \subset \operatorname{Ker}\{R(A_1 + B_1F_1 + G_1C_1)|_{V_2}\}. \tag{27}$$

In fact, let an arbitrary element $x \in \operatorname{Ker} R = V_1$. Then,

$$\begin{aligned} R(A_1 + B_1F_1 + G_1C_1)x &= R(A_1 + B_1F_0 + GC_1)x \quad (\text{by } (26)) \\ &\subset RV_1 \quad (\text{by } (24)) \\ &= \{0\}, \end{aligned}$$

showing that $x \in \operatorname{Ker}\{R(A_1 + B_1F_1 + G_1C_1)|_{V_2}\}$. Thus, (27) was proved.

Now, from (27) and Lemma 2.10, there exists an $N \in \mathbf{B}(\tilde{W})$ such that

$$NR = R(A_1 + B_1F_1 + G_1C_1)|_{V_2}. \tag{28}$$

We prove the following claim.

Claim 1. $R(A_i + B_jF_k)x = R(A_1 + B_1F_1)x$ *for all* $x \in V_2$ *and all* $i, j, k \in \mathbf{2}$.

To prove this claim, let an arbitrary element $x \in V_2$. Then, we obtain

$$\begin{aligned} &R(A_i + B_jF_k)x - R(A_1 + B_1F_1)x \\ &\quad = R(A_i - A_1)x + R(B_jF_k - B_1F_1)x \\ &\quad = R(A_i - A_1)x + R\{B_j(KC_k + F_0) - B_1(KC_1 + F_0)\}x \\ &\quad = R(A_i - A_1)x + R(B_jKC_k - B_1KC_1)x + R(B_j - B_1)F_0x \\ &\quad = R(A_i - A_1)x + R(B_j - B_1)KC_1x + R(B_j - B_1)F_0x \\ &\qquad\qquad (\text{by } C_1x = C_2x) \\ &\quad \subset RV_1 + RV_1 \quad (\text{by } (A_1 - A_2)V_2 \subset V_1 \text{ and } \operatorname{Im}(B_2 - B_1) \subset V_1) \\ &\quad = \{0\}, \end{aligned}$$

showing Claim 1.

Next, we prove the following claim.

Claim 2. $(MC_k + NR)x = R(A_i + B_jF_k)x$ for all $x \in V_2$ and all $i, j, k \in \mathbf{2}$.

To prove Claim 2, let an arbitrary element $x \in V_2$. Then, we obtain

$$\begin{aligned} (MC_k + NR)x &= (NR - RG_1C_k)x \\ &= \{R(A_1 + B_1F_1 + G_1C_1) - RG_1C_k\}x \quad (\text{by } (28)) \\ &= R(A_1 + B_1F_1)x + RG_1(C_1 - C_k)x \\ &= R(A_i + B_jF_k)x \text{ for all } i, j, k \in \mathbf{2} \\ &\qquad (\text{by } V_2 \subset \operatorname{Ker}(C_1 - C_2) \text{ and Claim 1}), \end{aligned}$$

showing Claim 2.

Finally, we prove the following claim.

Claim 3. $A^e_{ijk}V^e \in V^e$ for all $i, j, k \in \mathbf{2}$.

To prove Claim 3, let an arbitrary element $\left[\begin{smallmatrix} x \\ Rx \end{smallmatrix}\right] \in V^e$ $(x \in V_2)$.

Hence,

$$\begin{aligned}
A^e_{ijk}\begin{bmatrix} x \\ Rx \end{bmatrix} &= \begin{bmatrix} A_i + B_jKC_k & B_jL \\ MC_k & N \end{bmatrix}\begin{bmatrix} x \\ Rx \end{bmatrix} \\
&= \begin{bmatrix} (A_i + B_jKC_k)x + B_jLRx \\ MC_kx + NRx \end{bmatrix} \qquad (29) \\
&= \begin{bmatrix} \left(A_i + B_jKC_k + B_jF_0R^{\dagger}R\right)x \\ (MC_k + NR)x \end{bmatrix} \\
&= \begin{bmatrix} \{A_i + B_j(KC_k + F_0)\}x \\ (MC_k + NR)x \end{bmatrix} \quad \left(\text{by } R^{\dagger}Rx - x \in V_1 \subset \operatorname{Ker} F_0\right) \\
&= \begin{bmatrix} (A_i + B_jF_k)x \\ (MC_k + NR)x \end{bmatrix} \\
&= \begin{bmatrix} (A_i + B_jF_k)x \\ R(A_i + B_jF_k)x \end{bmatrix} \quad (\text{by Claim 2}) \\
&\in V^e \quad \text{for all } i, j, k \in \mathbf{2}, \qquad (30)
\end{aligned}$$

showing Claim 3. Thus, since $V^e \subset \mathrm{D}(A^e_{ijk})$ for all $i, j, k \in \mathbf{2}$, it follows from Lemma 2.9(ii) that V^e is $\{S_{A^e_{ijk}};\ i, j, k \in \mathbf{2}\}$-invariant subspace. This completes the proof of this lemma. ∎

PROPOSITION 2.13. *Suppose that system* (1) *satisfies* $A_0 := A_1 = A_2$ *and* $B_0 := B_1 = B_2$. *If a pair* (V_1, V_2) *of subspaces of* X *is an* $\{S(C_k, A_0, B_0);\ k \in \mathbf{2}\}$*-pair satisfying* $V_2 \subset \operatorname{Ker}(C_1 - C_2)$, *then there exist a compensator* (K, L, M, N) *on* $\tilde{W} := (V_2 \cap V_1^{\perp})$ *and an* $\{S_{A^e_{00k}}(t);\ k \in \mathbf{2}\}$*-invariant subspace* $V^e(\subset X^e)$ *satisfying* $V_1 = S_1$ *and* $V_2 = S_2$, *where* $X^e := X \oplus \tilde{W}$, *and* S_1 *and* S_2 *are defined in Lemma* 2.6.

Proof. Suppose that a pair (V_1, V_2) of subspaces is an $\{S(C_k, A_0, B_0);\ k \in \mathbf{2}\}$-pair satisfying $V_2 \subset \operatorname{Ker}(C_1 - C_2)$. Let $R\colon V_2 \to \tilde{W}$, $R^{\dagger} \in \mathbf{B}(\tilde{W}; V_2)$ and V^e be the same as in the proof of Proposition 2.12. Then, we have $V_1 = S_1$ and $V_2 = S_2$.

Moreover, it follows from Lemma 2.11 that there exists $G \in \mathbf{G}_s(V_1)$, $G_0(:= G_1 = G_2) \in \mathbf{B}(\mathbf{R}^p; X)$, $F_k \in \mathbf{F}_s(V_2)$, $F_0 \in \mathbf{B}(X; \mathbf{R}^m)$ and $K \in \mathbf{B}(\mathbf{R}^p; \mathbf{R}^m)$ such that

$$G = B_0 K + G_0, \quad \operatorname{Im} G_0 \subset V_2$$
$$F_k = KC_k + F_0, \quad \operatorname{Ker} F_0 \supset V_1 \quad \text{for all } k \in \mathbf{2}.$$

Since $S_{A_0 + B_0 F_k}(t) V_2 \subset V_2$ for all $t \geq 0$ and all $k \in \mathbf{2}$ and $\operatorname{Im} G_0 C_k \subset V_2$, it follows from Lemma 2.9(iv) that

$$S_{A_0 + B_0 F_k + G_0 C_k}(t) V_2 \subset V_2 \quad \text{for all } t \geq 0 \text{ and all } k \in \mathbf{2}. \tag{31}$$

Furthermore, since $S_{A_0 + GC_k}(t) V_1 \subset V_1$ for all $t \geq 0$ and all $k \in \mathbf{2}$ and $(B_0 F_k + G_0 C_k - GC_k)x = B_0 F_0 x = 0$ for all $x \in V_1$, it follows from Lemma 2.9(v) that

$$S_{A_0 + B_0 F_k + G_0 C_k}(t) x = S_{A_0 + GC_k}(t) x$$
$$\text{for all } t \geq 0, \text{ all } k \in \mathbf{2} \text{ and all } x \in V_1. \tag{32}$$

By (31) we can define the following bounded linear operator on $\tilde{W}$.

$$S_N(t) := RS_{A_0 + B_0 F_k + G_0 C_1}(t) R^{\dagger}$$

Furthermore, since $(R^{\dagger} Rx - x) \in V_1 = \operatorname{Ker} R$ for all $x \in V_2$, it follows from (32) that

$$RS_{A_0 + B_0 F_1 + G_0 C_1}(t) R^{\dagger} Rx = RS_{A_0 + B_0 F_1 + G_0 C_1}(t) x$$
$$\text{for all } t \geq 0, \text{ and all } x \in V_2. \tag{33}$$

In fact,

$$\begin{aligned} R\left[S_{A_0 + B_0 F_1 + G_0 C_1}(t)(R^{\dagger} Rx - x) \right] &= R\left[S_{A_0 + GC_1}(t)(R^{\dagger} Rx - x) \right] \quad (\text{by } (32)) \\ &\subset RV_1 \\ &= \{0\}. \end{aligned}$$

Now, we prove $\{S_N(t);\ t \geq 0\}$ is a C_0-semigroup on $\tilde{W}$. First, we have

$$\begin{aligned} S_N(t) Rx &= RS_{A_0 + B_0 F_1 + G_0 C_1}(t) R^{\dagger} Rx \\ &= RS_{A_0 + B_0 F_1 + G_0 C_1}(t) x \quad \text{for all } x \in V_2 \ (\text{by } (33)). \end{aligned} \tag{34}$$

Hence, for arbitrary $w \in \tilde{W}$

$$
\begin{aligned}
S_N(t) S_N(s) w &= S_N(t) R S_{A_0+B_0F_1+G_0C_1}(s) R^\dagger w \\
&= R S_{A_0+B_0F_1+G_0C_1}(t) S_{A_0+B_0F_1+G_0C_1}(s) R^\dagger w \quad (\text{by } (34)) \\
&= R S_{A_0+B_0F_1+G_0C_1}(t+s)) R^\dagger w \\
&= S_N(t+s) w.
\end{aligned}
$$

Thus, $\{S_N(t);\, t \geq 0\}$ is a C_0-semigroup on $\tilde{W}$. Define

$$
\begin{aligned}
M &:= -RG_0 \in \mathbf{B}(\mathbf{R}^p; \tilde{W}), \\
L &:= F_0 R^\dagger \in \mathbf{B}(\tilde{W}; \mathbf{R}^m), \\
N &:= \text{the generator of a } C_0\text{-semigroup } \{S_N(t);\, t \geq 0\}.
\end{aligned}
$$

Then, we claim the following assertions.

Claim 1. $S_{A_0+B_0F_k+G_0C_k}(t)x = S_{A_0+B_1F_1+G_1C_1}(t)x$ for all $t \geq 0$, all $k \in \mathbf{2}$ and all $x \in V_2$.

From (31) and Lemma 2.9(v) it suffices to show that

$$
\{B_0F_k + G_0C_k - (B_1F_1 + G_1C_1)\} V_2 = \{0\}. \tag{35}
$$

To prove (35) we choose an arbitrary $x \in V_2$. Then, we have

$$
\begin{aligned}
&\{B_0F_k + G_0C_k - (B_1F_1 + G_1C_1)\} x \\
&\quad = \{B_0(KC_k + F_0) + (G - B_0K)C_k \\
&\qquad - B_1(KC_1 + F_0) - (G - B_1K)C_1\} x \\
&\quad = (B_0 - B_1) F_0 x + G(C_k - C_1) x \\
&\quad = 0, \quad (\text{by } B_0 = B_1 \text{ and } V_2 \subset \operatorname{Ker}(C_2 - C_1))
\end{aligned}
$$

showing (35). Thus, the Claim 1 was proved. Next, we claim the following.

Claim 2.

$$
\begin{bmatrix} S_{A_0+B_0F_k+G_0C_k}(t) & 0 \\ 0 & S_N(t) \end{bmatrix} \begin{bmatrix} x \\ Rx \end{bmatrix} \in V^e \quad \text{for all } \begin{bmatrix} x \\ Rx \end{bmatrix} \in V^e \ (x \in V_2).
$$

In fact,

$$\begin{aligned}
\begin{bmatrix} S_{A_0+B_0F_k+G_0C_k}(t) & 0 \\ 0 & S_N(t) \end{bmatrix}\begin{bmatrix} x \\ Rx \end{bmatrix} &= \begin{bmatrix} S_{A_0+B_0F_k+G_0C_k}(t)x \\ RS_{A_0+B_0F_1+G_0C_1}(t)R^{\dagger}Rx \end{bmatrix} \\
&= \begin{bmatrix} S_{A_0+B_0F_k+G_0C_k}(t)x \\ RS_{A_0+B_0F_1+G_0C_1}(t)x \end{bmatrix} \quad (\text{by } (33)) \\
&= \begin{bmatrix} S_{A_0+B_0F_1+G_0C_1}(t)x \\ RS_{A_0+B_0F_1+G_0C_1}(t)x \end{bmatrix} \quad (\text{by Claim 1}) \\
&\in V^e. \qquad (36)
\end{aligned}$$

Noticing that

$$\begin{aligned}
A_{00k}^e &= \begin{bmatrix} A_0 + B_0kC_k & B_0L \\ MC_k & N \end{bmatrix} \\
&= \begin{bmatrix} A_0 + B_0F_k + G_0C_k & 0 \\ 0 & N \end{bmatrix} + \begin{bmatrix} -B_0F_0 - G_0C_k & B_0L \\ MC_k & 0 \end{bmatrix},
\end{aligned}$$

we prove $S_{A_{00k}^e}V^e \subset V^e$ for all $t \geq 0$ and all $k \in \mathbf{2}$. By Claim 2 and Lemma 2.9(iv) it suffices to show that

$$\begin{bmatrix} -B_0F_0 - G_0C_k & B_0L \\ MC_k & 0 \end{bmatrix} V^e \subset V^e. \qquad (37)$$

To prove (37) choose an arbitrary element $\left[\begin{smallmatrix} x \\ Rx \end{smallmatrix}\right] \in V^e (x \in V_2)$. Then, we have

$$\begin{aligned}
&\begin{bmatrix} -B_0F_0 - G_0C_k & B_0L \\ MC_k & 0 \end{bmatrix}\begin{bmatrix} x \\ Rx \end{bmatrix} \\
&\quad = \begin{bmatrix} -(B_0F_0 + G_0C_k)x + B_0LRx \\ MC_kx \end{bmatrix} \\
&\quad = \begin{bmatrix} -B_0(F_0 - LR)x - G_0C_kx \\ MC_kx \end{bmatrix}
\end{aligned}$$

$$= \begin{bmatrix} -B_0 F_0(x - R^\dagger Rx) - G_0 C_k x \\ MC_k x \end{bmatrix}$$

$$= \begin{bmatrix} -G_0 C_k x \\ -RG_0 C_k x \end{bmatrix} \quad \left(\text{by } x - R^\dagger Rx \in V_1 \subset \operatorname{Ker} F_0\right)$$

$$\in V^e \ (\text{by } \operatorname{Im} G_0 \subset V_2),$$

showing (37). Thus, $S_{A^e_{00k}} V^e \subset V^e$ for all $t \geq 0$ and all $k \in \mathbf{2}$. This completes the proof. ▮

3. ROBUST DISTURBANCE-REJECTION BY DYNAMIC COMPENSATOR

The linear control system to be considered is given by

$$\begin{aligned} \frac{d}{dt}x(t) &= Ax(t) + Bu(t) + E\xi(t), \qquad x(0) = x_0 \in X \\ y(t) &= Cx(t) \\ z(t) &= Dx(t) \end{aligned} \tag{38}$$

where $x(t) \in X$, $u(t) \in \mathbf{R}^m$, $y(t) \in \mathbf{R}^p$ and $z(t) \in \mathbf{R}^q$ are the state, the input, the measurement output and the controlled output, respectively. $\xi(\cdot)$ represents a disturbance which is a locally integrable function from $(0, \infty)$ to a Hilbert space Q (i.e., $\xi(\cdot) \in L_1^{\text{loc}}(0, \infty; Q)$).

It is assumed that operators A, B, C, D, and E are unknown, but they have the following convex combinations:

$$\begin{aligned} A &= \alpha A_1 + (1 - \alpha) A_2, \qquad B = \beta B_1 + (1 - \beta) B_2, \\ C &= \gamma C_1 + (1 - \gamma) C_2, \qquad D = \delta D_1 + (1 - \delta) D_2, \\ E &= \sigma E_1 + (1 - \sigma) E_2, \end{aligned} \tag{39}$$

where parameters $\alpha, \beta, \gamma, \delta, \sigma \in [0, 1]$ are unknown and operator A_i is the infinitesimal generator of a C_0-semigroup $\{S_{A_i}(t);\ t \geq 0\}$ on X, $B_i \in \mathbf{B}(\mathbf{R}^m; X)$, $C_i \in \mathbf{B}(X; \mathbf{R}^p)$, $D_i \in \mathbf{B}(X; \mathbf{R}^q)$ and $E_i \in \mathbf{B}(Q; X)$, and all these operators are known. Here, we note that, even if A_1 and A_2 are infinitesimal generators, $A = \alpha A_1 + (1 - \alpha) A_2 (\alpha \in [0, 1])$ may not be an infinitesimal generator. However, it is not difficult to see that if A_0 is the infinitesimal generator of a C_0-semigroup $\{S_{A_0}(t);\ t \geq 0\}$ on X, satisfying $\|S_{A_0}(t)\| \leq e^{\omega t}$ $(t \geq 0)$, then $A := aA_0$ $(0 < p \leq a \leq q$, p and q are given known constants) is also the infinitesimal generator of a C_0-semigroup $S_A(t);\ t \geq 0\}$ on X, satisfying $\|S_A(t)\| \leq e^{a\omega t}$ $(t \geq 0)$ e.g., see [5] p. 416).

Therefore, if $A_1 := pA_0$ and $A_2 := qA_0$ with $\mathrm{D}(A) = \mathrm{D}(A_0) = \mathrm{D}(A_1) = \mathrm{D}(A_2)$, then the operator $A = aA_0$ can be represented in the form $A = \alpha A_1 + (1 - \alpha)A_2$ for some $\alpha \in [0, 1]$. On the other hand, operators B, C, D, and E are always bounded linear operators.

If a compensator of the form (2) is applied to system (38), the resulting closed-loop system with the extended state space $X^e := X \oplus W$ is easily obtained as

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} x(t) \\ w(t) \end{bmatrix} &= \begin{bmatrix} A + BKC & BL \\ MC & N \end{bmatrix}\begin{bmatrix} x(t) \\ w(t) \end{bmatrix} + \begin{bmatrix} E \\ 0 \end{bmatrix}\xi(t), \\ z(t) &= \begin{bmatrix} D & 0 \end{bmatrix}\begin{bmatrix} x(t) \\ w(t) \end{bmatrix}. \end{aligned} \tag{40}$$

For convenience, we set

$$x^e(t) := \begin{bmatrix} x(t) \\ w(t) \end{bmatrix}, \quad A^e := \begin{bmatrix} A + BKC & BL \\ MC & N \end{bmatrix},$$

$$E^e := \begin{bmatrix} E \\ 0 \end{bmatrix} \quad \text{and} \quad D^e := \begin{bmatrix} D & 0 \end{bmatrix}.$$

Then, our robust disturbance-rejection problem with dynamic compensator is to find a compensator (K, L, M, N) of (2) such that

$$D^e \int_0^t S_{A^e}(t - \tau) E^e \xi(\tau)\, d\tau = 0$$

for all $\xi(\cdot) \in L_1^{\mathrm{loc}}(0, \infty; Q)$, all $t \geq 0$ and all parameters $\alpha, \beta, \gamma, \delta, \sigma \in [0, 1]$.

This problem can be formulated as follows.

Robust Disturbance-Rejection Problem with Dynamic Compensator (RDRPDC)

Given A_1, A_2, B_1, B_2, C_1, C_2, D_1, D_2, E_1 and E_2 of (39), find (if possible) a compensator (K, L, M, N) of (2) such that

$$\langle S_{A^e}(\cdot) | \mathrm{Im}\, E^e \rangle := \overline{L\Big(\bigcup_{t \geq 0} S_{A^e}(t)(\mathrm{Im}\, E^e) \Big)} \subset \mathrm{Ker}\, D^e$$

for all parameters $\alpha, \beta, \gamma, \delta, \sigma \in [0, 1]$, where $L(\Omega)$ and the over bar indicate the linear subspace generated by the set Ω and the closure in X^e, respectively.

The following theorem is an extension of the result of Ghosh [4] to infinite-dimensional systems.

THEOREM 3.1. *Suppose that system* (38) *with* (39) *satisfies* $\mathrm{D}(A_1) \subset \mathrm{D}(A_2)$. *If there exists an* $\{S(C_k, A_i, B_j);\ i, j, k \in \mathbf{2}\}$*-pair* (V_1, V_2) *such that* $\mathrm{Im}(B_1 - B_2) \subset V_1, V_2 \subset \{\mathrm{D}(A_1) \cap \ker(C_1 - C_2)\}$, $(A_1 - A_2)V_2 \subset V_1$ *and* $(\mathrm{Im}\, E_1 + \mathrm{Im}\, E_2) \subset V_1 \subset V_2 \subset (\mathrm{Ker}\, D_1 \cap \mathrm{Ker}\, D_2)$, *then the* RDRPDC *is solvable*.

Proof. Suppose that there exists an $\{S(C_k, A_i, B_j);\ i, j, k \in \mathbf{2}\}$-pair (V_1, V_2) satisfying all the stated conditions. Then, one has

$$\mathrm{Im}\, E \subset V_1 \subset V_2 \subset \mathrm{Ker}\, D$$

for all $\delta, \sigma \in [0,1]$. Further, by Proposition 2.12 there exists a compensator (K, L, M, N) on $\tilde{W} := (V_2 \cap V_1^{\perp})$ and $\{S_{A^e_{ijk}}(t);\ i, j, k \in \mathbf{2}\}$-invariant subspace $V^e (\subset X^e)$ satisfying $V_1 = S_1$ and $V_2 = S_2$. Then, it remarks that $V^e \subset \mathrm{D}(A^e)$ for all $\alpha, \beta, \gamma \in [0,1]$ and $\mathrm{Im}\, E^e \subset V^e \subset \mathrm{Ker}\, D^e$.

Now, let $x^e \in V^e \subset \mathrm{D}(A^e)$ be an arbitrary element. Then, we have

$$\begin{aligned}
&\begin{bmatrix} A + BKC & BL \\ MC & N \end{bmatrix} x^e \\
&\quad = \begin{bmatrix} A_1 + B_1KC_1 & B_1L \\ MC_1 & N \end{bmatrix} x^e + (1-\alpha)\begin{bmatrix} A_2 - A_1 & 0 \\ 0 & 0 \end{bmatrix} x^e \\
&\qquad + (1-\beta)\begin{bmatrix} B_2KC_1 - B_1KC_1 & (B_2 - B_1)L \\ 0 & 0 \end{bmatrix} x^e \\
&\qquad + (1-\gamma)\begin{bmatrix} B_1KC_2 - B_1KC_1 & 0 \\ M(C_2 - C_1) & 0 \end{bmatrix} x^e \\
&\qquad + (1-\beta)(1-\gamma)\begin{bmatrix} B_1KC_1 + B_2KC_2 - B_1KC_2 - B_2KC_1 & 0 \\ 0 & 0 \end{bmatrix} x^e \\
&\quad = A^e_{111} x^e + (1-\alpha)\left(A^e_{2jk} - A^e_{1jk}\right) x^e + (1-\beta)\left(A^e_{i21} - A^e_{i11}\right) x^e \\
&\qquad + (1-\gamma)\left(A^e_{i12} - A^e_{i11}\right) x^e \\
&\qquad + (1-\beta)(1-\gamma)\left(A^e_{i11} + A^e_{i22} - A^e_{i12} - A^e_{i21}\right)^e \\
&\quad \in V^e \quad \text{for all } \alpha, \beta, \gamma \in [0,1],
\end{aligned}$$

which implies

$$A^e V^e \subset V^e$$

for all $\alpha, \beta, \gamma \in [0,1]$. It follows from Lemma 2.9(ii) that V^e is $S_{A^e}(t)$-invariant for all $t \geq 0$ and all $\alpha, \beta, \gamma \in [0,1]$. Therefore, the following relations can be easily obtained

$$\begin{aligned}\langle S_{A^e}(\cdot)|\mathrm{Im}\, E^e\rangle &\subset \langle S_{A^e}(\cdot)|V^e\rangle \\ &= V^e \\ &\subset \mathrm{Ker}\, D^e\end{aligned}$$

for all $\alpha, \beta, \gamma, \delta, \sigma \in [0,1]$. Thus, the RDRPDC is solvable. This completes the proof. ∎

COROLLARY 3.2. *Suppose that system* (38) *with* (39) *satisfies* $\mathrm{D}(A_1) \subset \mathrm{D}(A_2)$. *If there exists an* $\{S(C_k, A_i, B_j);\ i, j, k \in \mathbf{2}\}$*-pair* (V_1, V_2) *satisfying all the stated conditions of Theorem* 3.1 *and* $\dim(V_2 \cap V_1^{\perp}) < \infty$, *then the* RDRPDC *with finite-dimensional dynamic compensator is solvable.*

Proof. The proof follows from the proof of Theorem 3.1. ∎

THEOREM 3.3. *Suppose that system* (38) *with* (39) *satisfies* $A_0 := A_1 = A_2$ *and* $B_0 := B_1 = B_2$. *If there exists an* $\{S(C_k, A_0, B_0);\ k \in \mathbf{2}\}$*-pair* (V_1, V_2) *such that* $V_2 \subset \mathrm{Ker}(C_1 - C_2)$ *and* $(\mathrm{Im}\, E_1 + \mathrm{Im}\, E_2) \subset V_1 \subset V_2 \subset (\mathrm{Ker}\, D_1 \cap \mathrm{Ker}\, D_2)$, *then the* RDRPDC *is solvable.*

Proof. Suppose that there exists an $\{S(C_k, A_0, B_0);\ k \in \mathbf{2}\}$-pair (V_1, V_2) satisfying the conditions given above. It follows from Proposition 2.13 that there exist a compensator (K, L, M, N) on $\tilde{W} := (V_2 \cap V_1^{\perp})$ and $\{S_{A^e_{00k}}(t);\ k \in \mathbf{2}\}$-invariant subspace $V^e(\subset X^e)$ satisfying $V_1 = S_1$ and $V_2 = S_2$. It is first noted that

$$\mathrm{Im}\, E^e \subset V^e \subset \mathrm{Ker}\, D^e.$$

for all $\delta, \sigma \in [0,1]$. Further, since

$$\begin{aligned}&\left(\begin{bmatrix} B_0KC & B_0L \\ MC & 0\end{bmatrix} - \begin{bmatrix} B_0KC_1 & B_0L \\ MC_1 & 0\end{bmatrix}\right)(V^e \cap \mathrm{D}(A^e)) \\ &\quad= (A^e - A^e_{001})(V^e \cap \mathrm{D}(A^e)) \\ &\quad= ((\gamma - 1)A^e_{001} + (1-\gamma)A^e_{002})(V^e \cap \mathrm{D}(A^e)) \\ &\quad\subset V^e,\end{aligned}$$

it follows from Lemma 2.9(iv) that

$$S_{A^e}(t)V^e \subset V^e$$

for all $t \geq 0$ and all $\gamma \in [0, 1]$, and moreover, the following relations can be easily obtained.

$$\begin{aligned} \langle S_{A^e}(\cdot)|\mathrm{Im}\, E^e\rangle &\subset \langle S_{A^e}(\cdot)|V^e\rangle \\ &= V^e \\ &\subset \mathrm{Ker}\, D^e \end{aligned}$$

for all $\gamma, \delta, \sigma \in [0, 1]$. Thus, the RDRPDC is solvable. This completes the proof. ∎

COROLLARY 3.4. *Suppose that system* (38) *with* (39) *satisfies* $A_0 := A_1 = A_2$ *and* $B_0 := B_1 = B_2$. *If there exists an* $\{S(C_k, A_0, B_0);\ k \in \mathbf{2}\}$*-pair* (V_1, V_2) *satisfying all the stated conditions of Theorem* 3.3 *and* $\dim(V_2 \cap V_1^{\perp}) < \infty$, *then the* RDRPDC *with finite-dimensional dynamic compensator is solvable.*

Proof. The proof follows from the proof of Theorem 3.3. ∎

4. CONCLUSIONS

In this paper, an infinite-dimensional version of the simultaneous (C, A, B)-pair was introduced, and its interesting properties were investigated. Further, by using these results the infinite-dimensional version of robust disturbance-rejection problem with dynamic compensator was formulated, and its solvability conditions were obtained under certain assumptions.

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